

On Rainbow- k -Connectivity of Random Graphs*

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December 15, 2010

Abstract

A path in an edge-colored graph is called a *rainbow path* if the edges on it have distinct colors. For $k \geq 1$, the *rainbow- k -connectivity* of a graph G , denoted $rc_k(G)$, is the minimum number of colors required to color the edges of G in such a way that every two distinct vertices are connected by at least k internally vertex-disjoint rainbow paths. In this paper, we study rainbow- k -connectivity in the setting of random graphs. We show that for every fixed integer $d \geq 2$ and every $k \leq O(\log n)$, $p = (\log n)^{1/d}/n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leq d$. This substantially generalizes a result in [Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, On rainbow connection, Electron. J. Comb., 15, 2008], stating that $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc_1(G(n, p)) \leq 2$. As a by-product, we obtain a polynomial-time algorithm that makes $G(n, p)$ rainbow- k -connected using at most one more than the optimal number of colors with probability $1 - o(1)$, for all $k \leq O(\log n)$ and $p = n^{-\epsilon(1 \pm o(1))}$ for any constant $\epsilon \in [0, 1)$.

1 Introduction

All graphs considered in this paper are finite, simple, undirected and contain at least 2 vertices. We follow the notation and terminology of [3]. The following notion of *rainbow- k -connectivity* was proposed by Chartrand et al. [8, 9] as a strengthening of the canonical connectivity concept in graphs. Given an edge-colored graph G , a path in G is called a *rainbow path* if its edges have distinct colors. For an integer $k \geq 1$, an edge-colored graph is called *rainbow- k -connected* if any two different vertices of G are connected by at least k internally vertex-disjoint rainbow paths. The *rainbow- k -connectivity* of G , denoted by $rc_k(G)$, is the minimum number of colors required to color the edges of G to make it rainbow- k -connected. Note that such coloring does not exist if G is not k -vertex-connected, in which case we simply let $rc_k(G) = \infty$. When $k = 1$ it is alternatively called *rainbow-connectivity* or *rainbow connection number* in literature, and is conventionally written as $rc(G)$ with the subscript k dropped.

Besides its theoretical interest as being a natural combinatorial concept, rainbow connectivity also finds applications in networking and secure message transmitting [6, 11, 15]. The following motivation is given in [6]: Suppose we want to route messages in a cellular network such that each link on the route between two vertices is assigned with a distinct channel. Then the minimum number of used channels is exactly the rainbow-connectivity of the underlying graph.

*This work was supported in part by the National Basic Research Program of China Grant 2011CBA00300, 2011CBA00301, and the National Natural Science Foundation of China Grant 61033001, 61061130540, 61073174.

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Some easy observations regarding rainbow- k -connectivity include that $rc_k(G) = 1$ if and only if $k = 1$ and G is a clique, that $rc(G) \leq n - 1$ for all connected G , and that $rc(G) = n - 1$ if and only if G is a tree, where n is the number of vertices in G . Chartrand et al. [8] determined the rainbow-connectivity of several special classes of graphs, including complete multipartite graphs. In [9] they investigated rainbow- k -connectivity in complete graphs and regular complete bipartite graphs. The extremal graph-theoretic aspect of rainbow-connectivity was studied by Caro et al. [5], who proved that $rc(G) = O_\delta(n \log \delta / \delta)$ with δ being the minimum degree of G . This tradeoff was later improved to $rc(G) < 20n/\delta$ by Krivelevich and Yuster [13], and was recently shown to be $rc(G) \leq 3n/(\delta + 1) + 3$ by Chandran et al. [7] which is essentially tight. Chakraborty et al. [6] studied the computational complexity perspective of this notion, proving among other results that given a graph G deciding whether $rc(G) = 2$ is NP-complete.

Another important setting that has been extensively explored for studying various graph concepts is the Erdős-Rényi random graph model $G(n, p)$ [10], in which each of the $\binom{n}{2}$ pairs of vertices appears as an edge with probability p independently from other pairs. We say an event \mathcal{E} happens *almost surely* if the probability that it happens approaches 1 as $n \rightarrow \infty$, i.e., $\Pr[\mathcal{E}] = 1 - o_n(1)$. We will always assume that n is the variable that tends to infinity, and thus omit the subscript n from the asymptotic notations. For a graph property P , a function $p(n)$ is called a *threshold function* of P if:

- for every $r(n) = \omega(p(n))$, $G(n, r(n))$ almost surely satisfies P ; and
- for every $r'(n) = o(p(n))$, $G(n, r'(n))$ almost surely does not satisfy P .

Furthermore, $p(n)$ is called a *sharp threshold function* of P if there exist two positive constants c and C such that:

- for every $r(n) \geq C \cdot p(n)$, $G(n, r(n))$ almost surely satisfies P ; and
- for every $r'(n) \leq c \cdot p(n)$, $G(n, r'(n))$ almost surely does not satisfy P .

Clearly a sharp threshold function of a graph property is also a threshold function of it; yet the converse may not hold, e.g., the property of containing a triangle [2].

It is known that every non-trivial monotone graph property possesses a threshold function [4, 12]. Obviously for every k, d , the property $rc_k(G) \leq d$ is monotone, and thus has a threshold. Caro et al. [5] proved that $p = \sqrt{\log n / n}$ is a sharp threshold function for the property $rc_1(G(n, p)) \leq 2$. In this paper, we significantly extend their result by establishing sharp thresholds for the property $rc_k(G(n, p)) \leq d$ for all constants d and logarithmically increasing k . Our main theorem is as follows.

Theorem 1. *Let $d \geq 2$ be a fixed integer and $k = k(n) \leq O(\log n)$. Then $p = (\log n)^{1/d} / n^{(d-1)/d}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leq d$.*

We also investigate rainbow- k -connectivity from the algorithmic point of view. The NP-hardness of determining $rc(G)$ is shown by Chakraborty et al. [6]. We show that the problem (even the search version) becomes easy in random graphs, by designing an algorithm for coloring random graphs to make it rainbow- k -connected with near-optimal number of colors.

Theorem 2. *For any constant $\epsilon \in [0, 1)$, $p = n^{-\epsilon(1 \pm o(1))}$ and $k \leq O(\log n)$, there is a randomized polynomial-time algorithm that, with probability $1 - o(1)$, makes $G(n, p)$ rainbow- k -connected using at most one more than the optimal number of colors, where the probability is taken over both the randomness of $G(n, p)$ and that of the algorithm.*

Our result is quite strong, since almost all natural edge probability functions p encountered in various scenarios satisfy $p = n^{-\epsilon(1 \pm o(1))}$ for some $\epsilon > 0$. Note that $G(n, n^{-\epsilon})$ is almost surely disconnected when $\epsilon > 1$ [10], which makes the problem become trivial. We therefore ignore these cases.

In Section 2 we present the proof of Theorem 1, and in Section 3 we show the correctness of Theorem 2.

2 Threshold of Rainbow- k -Connectivity

This section is devoted to proving Theorem 1. Throughout the paper “ln” denotes the natural logarithm, and “log” denotes the logarithm to the base 2. Hereafter we assume $d \geq 2$ is a fixed integer, $c_0 \geq 1$ a positive constant, and $k = k(n) \leq c_0 \log n$ for all sufficiently large n . To establish a sharp threshold function for a graph property the proof should be two-fold. We first show the easy direction.

Theorem 3. $rc_k(G(n, (\ln n)^{1/d}/n^{(d-1)/d})) \geq d+1$ almost surely holds.

We need the following fact proved by Bollobás [1].

Lemma 1 (Restatement of part of Theorem 6 in [1]). *Let c be a positive constant and $d \geq 2$ a fixed integer. Let $p' = (\ln(n^2/c))^{1/d}/n^{(d-1)/d}$. Then,*

$$\lim_{n \rightarrow \infty} \Pr[G(n, p') \text{ has diameter at most } d] = e^{-c/2}.$$

of Theorem 3. Fix an arbitrary $\epsilon > 0$ and choose a constant $c > 0$ so that $e^{-c/2} < \epsilon/2$. Let $p' = (\ln(n^2/c))^{1/d}/n^{(d-1)/d}$ and $p = (\ln n)^{1/d}/n^{(d-1)/d}$. Clearly $p \leq p'$ for all $n > c$.

By Lemma 1 and the definition of limits, there exists an $N_1 > 0$ such that for all $n > N_1$, $\Pr[G(n, p') \text{ has diameter at most } d] < e^{-c/2} + \epsilon/2 < \epsilon$, by our choice of c . Thus, for every $n > \max\{c, N_1\}$,

$$\Pr[G(n, p) \text{ has diameter at most } d] \leq \Pr[G(n, p') \text{ has diameter at most } d] < \epsilon.$$

Due to the arbitrariness of ϵ , this implies that the probability of $G(n, p)$ having diameter at most d is $o(1)$. This completes the proof of Theorem 3, since the rainbow- k -connectivity of a graph is at least as large as its diameter. \square

We are left with the other direction stated below. Fix $C = 2^{20} \cdot c_0$.

Theorem 4. $rc_k(G(n, C(\log n)^{1/d}/n^{(d-1)/d})) \leq d$ almost surely holds.

The key component of our proof of Theorem 4 is the following theorem.

Theorem 5. *With probability at least $1 - n^{-\Omega(1)}$, every two different vertices of $G(n, C(\log n)^{1/d}/n^{(d-1)/d})$ are connected by at least $2^{10d} c_0 \log n$ internally vertex-disjoint paths of length exactly d .*

Before demonstrating Theorem 5, we show how Theorem 4 follows from it.

of Theorem 4. Let G be an instance of $G(n, C(\log n)^{1/d}/n^{(d-1)/d})$ for which the condition in Theorem 5 holds; that is, every two different vertices of G have at least $c_1 \log n$ internally vertex-disjoint paths of length d connecting them, where $c_1 := 2^{10d}c_0$. To establish Theorem 4 it suffices to prove that $rc_k(G) \leq d$ for every such G , since by Theorem 5 the condition holds with probability at least $1 - n^{-\Omega(1)} = 1 - o(1)$.

Let $S = \{1, 2, \dots, d\}$ be a set of d distinct colors. We randomly color the edges of G with colors from S . Fix two distinct vertices u and v . Let P be a path of length d connecting u and v . The probability that P becomes a rainbow path under the random coloring is

$$q := d!/d^d \geq (d/e)^d/d^d \geq 4^{-d},$$

by Stirling's formula. Since there are at least $c_1 \log n$ such paths and they are all edge-disjoint, we can upper-bound the probability that at most $k-1$ of them become rainbow paths by

$$\begin{aligned} & \binom{c_1 \log n}{k-1} (1-q)^{c_1 \log n - (k-1)} \\ & \leq \binom{c_1 \log n}{c_0 \log n} (1-4^{-d})^{(c_1-c_0) \log n} \\ & \leq 2^{c_1 \log n \cdot H(c_0/c_1)} \cdot 2^{-4^{-d}(c_1-c_0) \log n} \\ & = n^{-(4^{-d}(c_1-c_0) - c_1 \cdot H(c_0/c_1))}, \end{aligned}$$

where the second inequality follows from the fact that

$$\binom{m}{\alpha m} \leq 2^{m \cdot H(\alpha)}$$

for all constants $\alpha \in (0, 1)$ and sufficiently large m , H being the binary entropy function defined as

$$H(\epsilon) = \epsilon \log(1/\epsilon) + (1-\epsilon) \log(1/(1-\epsilon)),$$

and that

$$1-x \leq e^{-x} \leq 2^{-x}, \text{ for all } x \geq 0.$$

It is easy to verify that $\log x \leq \sqrt{x}$ whenever $x \geq 100$. Also, since $1+x \leq e^x \leq 2^{2x}$, we have $\log(1+x) \leq 2x$ for all $x > -1$. Recalling that $c_1 = 2^{10d}c_0 > 200c_0$, we get

$$\begin{aligned} H(c_0/c_1) &= (c_0/c_1) \log(c_1/c_0) + (1-c_0/c_1) \log(1+c_0/(c_1-c_0)) \\ &\leq (c_0/c_1) \sqrt{c_1/c_0} + (1-c_0/c_1) \cdot 2c_0/(c_1-c_0) \\ &= \sqrt{c_0/c_1} + 2c_0/c_1 \leq 3\sqrt{c_0/c_1}. \end{aligned}$$

We thus have

$$\begin{aligned} 4^{-d}(c_1-c_0) - c_1 \cdot H(c_0/c_1) &\geq 4^{-d}(c_1-c_0) - 3\sqrt{c_1 c_0} \\ &= 4^{-d}c_1(1-2^{-10d}) - 3\sqrt{2^{-10d} \cdot c_1^2} \\ &\geq 2^{-2d-1}c_1 - 2^{-5d+2}c_1 \\ &\geq c_1 \cdot 2^{-2d-2} \\ &= c_0 \cdot 2^{10d} \cdot 2^{-2d-2} > 100, \end{aligned}$$

since $c_0 \geq 1$ and $d \geq 2$. Therefore, the probability that there exist at most $k - 1$ rainbow paths between u and v is at most

$$n^{-(4^{-d}(c_1 - c_0) - c_1 \cdot H(c_0/c_1))} < n^{-100}.$$

By the Union Bound, with probability at least

$$1 - \binom{n}{2} n^{-100} \geq 1 - n^{-90},$$

every two distinct vertices of G have at least k internally vertex-disjoint rainbow paths connecting them. In particular, there exists a d -coloring of the edges of G under which G becomes k -rainbow-connected, implying that $rc_k(G) \leq d$. This concludes the proof of Theorem 4. \square

We now prove Theorem 5.

of Theorem 5. Let $p = C(\log n)^{1/d}/n^{(d-1)/d}$ where $C = 2^{20}c_0$. Let V be the set of all vertices in $G(n, p)$. For every $S \subseteq V$ and $u \in S$, let $\mathcal{A}(S, u)$ be the event that u is adjacent to at least $pn/10 (= Cn^{1/d}(\log n)^{1/d}/10)$ distinct vertices in $V \setminus S$. The following lemma is needed for our proof.

Lemma 2. *For every S, u such that $u \in S$ and $|S| \leq d \cdot (pn/10)^{d-1}$,*

$$\Pr[\mathcal{A}(S, u)] \geq 1 - 2^{-\Omega(n^{1/d})}.$$

Proof. Fix $S \subseteq V$ with $|S| \leq d \cdot (pn/10)^{d-1}$, and $u \in S$. We have

$$|V \setminus S| \geq n - d \cdot (pn/10)^{d-1} = n - d \cdot (C/10)^{d-1} n^{(d-1)/d} (\log n)^{(d-1)/d} \geq n/2,$$

for all sufficiently large n . Take T to be any subset of $V \setminus S$ of cardinality $n/2$. Let X be the random variable counting the number of neighbors of u inside T . It is obvious that X can be expressed as the sum of $n/2$ independent random variables, each of which taking 1 with probability p and 0 with probability $1 - p$. Thus $\mathbf{E}[X] = pn/2$. By the Chernoff-Hoeffding Bound (see e.g. Theorem 4.2 of [14]), we have

$$\Pr[X < (1 - 4/5)pn/2] \leq \exp(-(1/2)(4/5)^2(pn/2)) = 2^{-\Omega(n^{1/d})},$$

which gives precisely what we want. \square

We now continue the proof of Theorem 5. Fix $u, v \in V, u \neq v$. Let $S_0 = \{u\}$. A t -ary tree with a designated root is a tree whose non-leaf vertices all have exactly t children. Consider the following process of “choosing” a $(pn/10)$ -ary tree of depth $d - 1$ rooted at u :

1. Let $i \leftarrow 1$ and $S_i \leftarrow \emptyset$.
2. For every vertex $w \in S_{i-1}$ (in an arbitrary order), choose $pn/10$ distinct neighbors of w from the set $V \setminus (\{v\} \cup \bigcup_{j=0}^i S_j)$, and add them to S_i . (Note that S_i is updated every time after the processing of a vertex w , so that one vertex cannot be chosen and added to S_i more than once. This ensures that at the end of this step, $|S_i| = (pn/10)^i$.)
3. Let $i \leftarrow i + 1$. If $i \leq d - 1$ then go to Step 2, otherwise stop.

Of course the process may fail during Step 2, since with nonzero probability w will have no neighbor in $V \setminus (\{v\} \cup \bigcup_{j=0}^i S_j)$. (In fact, with nonzero probability the graph becomes empty.) However, noting that at any time during the process,

$$|\{v\} \cup \bigcup_{j=0}^i S_j| \leq 1 + \sum_{j=0}^{d-1} (pn/10)^j \leq d \cdot (pn/10)^{d-1}, \text{ for all sufficiently large } n,$$

we can deduce from Lemma 2 that every execution of Step 2 fails with probability at most $2^{-\Omega(n^{1/d})}$. Since Step 2 can be conducted for at most $d \cdot (pn/10)^{d-1}$ times, we obtain that, with probability at least

$$1 - d \cdot (pn/10)^{d-1} \cdot 2^{-\Omega(n^{1/d})} = 1 - 2^{-\Omega(n^{1/d})},$$

the process will successfully terminate. At the end of the process, the sets S_0, S_1, \dots, S_{d-1} naturally induces a $(pn/10)$ -ary tree T of depth $d-1$ rooted at u , with S_i being the collection of vertices in the i -th level. The number of leaves in T is exactly $|S_{d-1}| = (pn/10)^{d-1}$.

Now we assume that T has been successfully constructed. Let Y be a random variable denoting the number of neighbors of v inside S_{d-1} . (Recall that $v \notin S_{d-1}$.) It is clear that

$$\mathbf{E}[Y] = p \cdot |S_{d-1}| = p^d n^{d-1} / 10^{d-1} = 10 \cdot (C/10)^d \log n.$$

As before, using the Chernoff-Hoeffding Bound, we have

$$\Pr[Y < (C/10)^d \log n] \leq \exp(-(1/2)(9/10)^2 (C/10)^d \cdot 10 \log n) \leq n^{-10},$$

for our choice of C .

It is clear that each neighbor v' of v inside S_{d-1} induces a length- d path between u and v (by simply combining the path from u to v' in tree T and the edge (v', v)). The problem is that these paths may not be internally vertex-disjoint. We next address this issue.

For every $w \in S_1$, denote by T_w the subtree of T of depth $d-2$ rooted at w . Call these T_w *vice-trees*. Clearly every vice-tree contains $(pn/10)^{d-2}$ leaves.

The reason for defining such vice-trees is that, by simple observations, any two leaves of T that belong to different vice-trees must correspond to edge-disjoint root-to-leaf paths in T . Thus, to establish a large number of internally vertex-disjoint paths between u and v , it suffices to show that we can find many neighbors of v inside S_{d-1} that belong to distinct vice-trees.

For each vice-tree T_w , let \mathcal{B}_w be the event that v has at least $10d$ neighbors inside the set of leaves of T_w . Noting that T_w has exactly $(pn/10)^{d-2}$ leaves, we have

$$\begin{aligned} \Pr[\mathcal{B}_w] &\leq \binom{(pn/10)^{d-2}}{10d} p^{10d} \\ &= \binom{(Cn^{1/d}(\log n)^{1/d}/10)^{d-2}}{10d} \cdot \left(\frac{C(\log n)^{1/d}}{n^{(d-1)/d}} \right)^{10d} \\ &\leq \left((Cn^{1/d}(\log n)^{1/d}/10)^{d-2} \cdot \frac{C(\log n)^{1/d}}{n^{(d-1)/d}} \right)^{10d} \\ &= C^{10d} (C/10)^{10d(d-2)} (\log n)^{10(d-1)} n^{-10} \\ &\leq O(n^{-9}). \end{aligned}$$

By applying the Union Bound, we obtain

$$\Pr\left[\bigvee_{w \in S_1} \mathcal{B}_w\right] \leq (pn/10) \cdot O(n^{-9}) \leq O(n^{-7}).$$

Combined with previous results, we deduce that with probability at least

$$1 - 2^{-\Omega(n^{1/d})} - n^{-10} - O(n^{-7}) \geq 1 - O(n^{-6}),$$

the following three events simultaneously happen:

1. The tree T is successfully constructed.
2. v has at least $(C/10)^d \log n$ neighbors that are leaves of T .
3. Every vice-tree T_w contains at most $10d$ leaves that are neighbors of v .

When all these three events happen, we can choose $((C/10)^d/(10d)) \log n$ neighbors of v , every two of which come from different vice-trees. This immediately leads to $((C/10)^d/(10d)) \log n$ length- d internally vertex-disjoint paths between u and v , where, by our choice of $C = 2^{20}c_0$,

$$((C/10)^d/(10d)) \log n \geq 2^{10d}c_0 \log n.$$

Using the Union Bound again gives us that, with probability at least

$$1 - \binom{n}{2} \cdot O(n^{-6}) = 1 - n^{-\Omega(1)},$$

every two distinct vertices have at least $2^{10d}c_0 \log n$ internally vertex-disjoint paths of length d connecting them. The proof of Theorem 5 is thus completed. \square

3 Rainbow-coloring Random Graphs

In this section we prove Theorem 2.

of Theorem 2. First note that for every G with at least 2 vertices, $rc_k(G) = 1$ if and only if $k = 1$ and G is a clique, which can be easily checked. Thus, in the following we assume w.l.o.g. that $rc_k(G(n, p)) \geq 2$.

It is easy to see that there exists a (unique) integer $d \geq 2$ such that $(d-2)/(d-1) \leq \epsilon < (d-1)/d$. We have $p = \omega((\log n)^{1/d}/n^{(d-1)/d})$, which, by Theorem 4, implies that $rc_k(G(n, p)) \leq d$ almost surely holds. Moreover, a scrutiny into the proof of Theorem 5 tells us that for such p , a random coloring of $G(n, p)$ using d colors will make it rainbow- k -connected with probability $1 - n^{-\Omega(1)}$. Thus, it suffices for us to show that with probability $1 - o(1)$, $rc_k(G(n, p)) \geq d - 1$. This is trivial for $d \leq 3$, since we have assumed that $rc_k(G(n, p)) \geq 2$. When $d \geq 4$, we have $p = o((\log n)^{1/(d-2)}/n^{(d-3)/(d-2)})$. Due to Theorem 3, $G(n, p)$ with such p almost surely satisfies $rc_k(G(n, p)) \geq d - 1$.

Hence, we have shown that with probability $1 - o(1)$, a random coloring with d colors will make $G(n, p)$ rainbow- k -connected and the number of colors used is at most one more than the optimum, where the probability is taken over both the randomness of $G(n, p)$ and that of the algorithm. This completes the whole proof. \square

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